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ALGEBRAS WITH AFFINE FIBRES OVER AN EXCELLENT RING

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We investigate an algebra R over an excellent ring D such that the fibre ring $R \otimes_D k(p)$ is finitely generated over $k(p)$ for every prime ideal p of D. This class of rings is studied in connection with pseudopolynomial rings, and we give a sufficient condition for R to be finitely generated over D.

Introduction

Let D be a noetherian ring. In this paper we are interested in a D-algebra R such that the fibre ring $R \otimes_D k(p)$ is finitely generated over $k(p)$ for every prime ideal p of D, where $k(p)$ stands for the residue field D_p/pD_p. If R is finitely generated over D then R clearly has this property, however the converse does not hold in general. It is thus natural to ask what conditions on a D-algebra R having this property imply that R is finitely generated over D. This problem is closely related to the theory of pseudopolynomial rings. Recall that R is said to be a pseudopolynomial D-algebra in n variables if each fibre ring $R \otimes_D k(p)$ is isomorphic to a polynomial ring in n variables over $k(p)$. This notion was introduced by Asanuma [1] and he determined the stable structure of a flat pseudopolynomial D-algebra R under the assumption that R is finitely generated over D (cf. [1, Theorem 3.4]). However this additional assumption is not needed when D is a discrete valuation ring (cf. [1, Theorem 3.1]), and in view of this result it seems plausible that we can remove it in general under a moderate assumption on D. This is a motivation of our consideration and the purpose of the present paper is to prove the following:

Theorem. *Let D be an excellent normal semi-local domain and let R be a flat D-algebra satisfying the following conditions*:

(i) *$R \otimes_D k(p)$ is a normal affine domain over $k(p)$ for every prime ideal p of D*;

(ii) *$\mathrm{tr.deg}_{k(p)} R \otimes_D k(p)$ is constant independent of p.*

Then R is a normal domain which is finitely generated over D.

As a corollary to this theorem we prove that if D is a noetherian semi-local ring and if R is a flat pseudopolynomial D-algebra, then R is finitely generated over D (cf. Corollary 4.6). Consequently, in the case where D is a noetherian semi-local ring, we can drop the above additional assumption from Asanuma's theorem.

The paper consists of four sections. In the first section we summarize some elementary lemmas which we need in later sections.

Sections 2 and 3 are devoted to preliminary investigations of an algebra R over a noetherian ring D which satisfies the conditions (i) and (ii) above; in the second section we prove some basic properties of R and in the third section we deal with an ideal-adic completion of R. The results, especially Lemmas 3.2 and 3.3, play an important role in Section 4.

After these preparatory considerations, we give a proof of our theorem in the final section. The main part of the proof is divided into two steps. The first step is to show that R_P is essentially of finite type over D for every prime ideal P of R (cf. Lemma 4.1) and the second step is to show that there exists a non-zero element x of R such that $R[x^{-1}]$ is finitely generated over D (cf. Lemma 4.2). Then, by making use of [5, Theorem 2.20], we complete the proof of the theorem stated above.

Notation and convention

In this paper a ring means a commutative ring with identity. For a ring D, we denote by $\mathrm{Max}(D)$ the set of all maximal ideals of D. When D is a semi-local ring with Jacobson radical J, we denote by $\hat{D}$ or $D^{\wedge}$ the J-adic completion of D. Let R be a D-algebra with structure homomorphism $f: D \to R$. Then R is said to be a D-algebra of finite type if R is finitely generated over $f(D)$. We call R an affine ring over D if f is injective and R is a D-algebra of finite type. For a prime ideal p of D, we abbreviate $R \otimes_D D_p$ as R_p. The other notation and terminology are essentially the same as those in [3]. In particular, the terminology 'normal' is used in accordance with [3, p. 116].

1. Preliminary lemmas

Throughout this section we assume that D is a noetherian ring and R is a D-algebra.

Lemma 1.1. (i) *Let $q_1, \ldots, q_r$ be ideals of D such that $q_1R \cap \cdots \cap q_rR$ is a finitely generated nilpotent ideal of R. Then R is a D-algebra of finite type if and only if $R \otimes_D D/q_i$ is a D/q_i-algebra of finite type for each $i = 1, \ldots, r$.*

(ii) *Let $\tilde{D}$ be a faithfully flat D-algebra. Then R is a D-algebra of finite type if and only if $R \otimes_D \tilde{D}$ is a $\tilde{D}$-algebra of finite type.*

(iii) *Suppose that D is a semi-local ring. Then R is a D-algebra of finite type if and only if R_m is a D_m-algebra of finite type for every maximal ideal m of D.*

Proof. In each case the necessity is obvious, and we give a proof only for the sufficiency, i.e., the 'if' part of the assertion.

(i) Let $\bar{R}=R/q_1R\times\cdots\times R/q_rR$ and let $I=q_1R\cap\cdots\cap q_rR$. Then there exists a natural injection $f: R/I\to\bar{R}$, and we may regard R/I as a subring of $\bar{R}$. Notice that $\bar{R}$ is a D-algebra of finite type because each $R/q_iR\cong R\otimes_D D/q_i$ is a D/q_i-algebra of finite type. Since $\bar{R}$ is a finite R/I-module, it then follows from [4, Lemma 3.1] that R/I is also a D-algebra of finite type. Hence there exists a subring A of R such that A is a D-algebra of finite type and that $\psi(A)=\psi(R)$, where ψ denotes the natural map $R\to R/I$. Then $R=A+I$ and hence, letting $x_1,\dots,x_s$ be the generators of I, $R=A[x_1,\dots,x_s]+I^n$ for every positive integer n. Since I is nilpotent, this implies that $R=A[x_1,\dots,x_s]$, and R is a D-algebra of finite type.

(ii) Let $y_1,\dots,y_t$ be elements of R such that $y_1\otimes 1,\dots,y_t\otimes 1$ generate $R\otimes_D\tilde{D}$ over $\tilde{D}$ and let B be the subring of R generated by $y_1,\dots,y_t$ over D. Then we have $B\otimes_D\tilde{D}=R\otimes_D\tilde{D}$, so that $B=R$ since $\tilde{D}$ is faithfully flat over D. Thus the assertion is verified.

(iii) By assumption there exists a subring C of R such that C is a D-algebra of finite type and that $C_m=R_m$ for every $m\in\operatorname{Max}(D)$. As is easily seen, the latter condition of C implies that $C=R$, which accomplishes the proof. □

Lemma 1.2. *Suppose that R is an integral domain and D is contained in R.*

(i) *If there exists a non-zero prime element x of R such that $R[x^{-1}]$ is normal, then R is normal.*

(ii) *Assume that R is flat over D and that there exists a non-zero element a of D such that $R[a^{-1}]$ is normal. If R_p is normal for every $p\in\operatorname{Ass}(D/aD)$, then R is normal.*

Proof. (i) Let R' be the integral closure of R in $Q(R)$. Then $R[x^{-1}]=R'[x^{-1}]$ since $R[x^{-1}]$ is normal. Hence, for an arbitrary element z of R', there is some positive integer n such that $x^nz\in R$. Then, letting p' be a prime ideal of R' lying over xR, we have $x^nz\in p'\cap R=xR$, which yields $x^{n-1}z\in R$ because R is an integral domain. We can repeat this argument to conclude that $z\in R$. Hence $R=R'$, and the assertion is verified.

(ii) Let $\Delta=\operatorname{Ass}(D/aD)$. Then we have $D=D[a^{-1}]\cap(\bigcap_{p\in\Delta}D_p)$, and so

$$\begin{aligned} R = R\otimes_D D &= R\otimes_D\left(D[a^{-1}]\cap\left(\bigcap_{p\in\Delta}D_p\right)\right)\\ &=(R\otimes_D D[a^{-1}])\cap\left(\bigcap_{p\in\Delta}R\otimes_D D_p\right)\\ &=R[a^{-1}]\cap\left(\bigcap_{p\in\Delta}R_p\right)\end{aligned}$$

because R is flat over D and Δ is a finite set. This shows that R is an intersection of normal domains, and therefore R is normal. □

In the remaining part of this section we fix an ideal m of D and, for a D-module M, we denote by M^* the m-adic completion of M.

Lemma 1.3. *Assume that R is flat over D. Then*:

(i) *R^* is flat over D.*

(ii) *For an ideal q of D, we have $(qR)^* \cong qR^*$ and $(R/qR)^* \cong R^*/qR^*$.*

Furthermore, if R/mR is noetherian, then:

(iii) *R^* is noetherian.*

(iv) $\dim R^* = \sup\{\dim (R_M)^\wedge \mid M \in \mathrm{Max}(R)$ *and* $mR \subset M\}$.

(v) $\dim (R_P)^\wedge = \dim (R_P)^*$ *for* $P \in \mathrm{Spec}(R)$ *with* $mR \subset P$.

Proof. First of all, we prove the following two claims:

Claim 1. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite D-modules. Then the induced sequence $0 \to (L \otimes_D R)^* \to (M \otimes_D R)^* \to (N \otimes_D R)^* \to 0$ is an exact sequence of R^*-modules.

Claim 2. Let M be a finite D-module. Then $(M \otimes_D R)^* \cong M \otimes_D R^*$.

For simplicity, we let $M_R = M \otimes_D R$ for a D-module M. In order to prove Claim 1, it suffices to show that the induced topology of L_R defined by $\{m^n M_R \cap L_R\}_{n=1,2,\dots}$ coincides with the m-adic topology of L_R. Notice that

$$m^n M_R = m^n(M \otimes_D R) = m^n M \otimes_D R$$

and

$$m^n M_R \cap L_R = (m^n M \otimes_D R) \cap (L \otimes_D R) = (m^n M \cap L) \otimes_D R$$

because R is flat over D. Since D is noetherian and M is a finite D-module, it follows from the Artin–Rees Lemma that there is a positive integer n' such that $m^{n'+k} M \cap L = m^k(m^{n'} M \cap L)$ for any non-negative integer k. Then, for any $k \geq 0$, we have

$$\begin{aligned} m^{n'+k} M_R \cap L_R &= (m^k(m^{n'} M \cap L)) \otimes_D R \\ &= m^k((m^{n'} M \otimes_D R) \cap (L \otimes_D R)) \\ &= m^k(m^{n'} M_R \cap L_R). \end{aligned}$$

This shows that the induced topology of L_R is the same thing as the m-adic topology of L_R, and Claim 1 is verified. Claim 2 follows from Claim 1 by using a standard argument (cf. the proof of Theorem 55 in [3], for example), and we omit details. We now proceed to the proof of the assertions.

(i) This is an easy consequence of Claims 1 and 2 above.

(ii) By (i) and Claim 2, we have $(qR)^* \cong (q \otimes_D R)^* \cong q \otimes_D R^* \cong qR^*$. It follows from this and Claim 1 that the exact sequence $0 \to q \to D \to D/q \to 0$ induces an exact sequence $0 \to qR^* \to R^* \to (R/qR)^* \to 0$. Hence $(R/qR)^* \cong R^*/qR^*$, as desired.

(iii) By (ii), we get $R^*/mR^* \cong (R/mR)^* = R/mR$. The assertion then follows from [7, Chapter VIII, §3, Corollary 4].

(iv) For any $M \in \mathrm{Max}(R)$ with $mR \subset M$, we have $(R_M)\hat{} \cong (R^*_{M^*})\hat{}$ because $R/M^n \cong R^*/(M^*)^n$ for every positive integer n. Hence, by virtue of (iii), we obtain

$$\mathrm{ht}(M^*) = \dim R^*_{M^*} = \dim (R^*_{M^*})\hat{} = \dim (R_M)\hat{}.$$

Since $\mathrm{Max}(R^*) = \{M^* \mid M \in \mathrm{Max}(R) \text{ and } mR \subset M\}$, the assertion follows from this.

(v) Let P be a prime ideal of R such that $mR \subset P$. Then, by assumption, R_P is flat over D and $R_P/mR_P \cong (R/mR)_P$ is noetherian. Hence, noting that R_P is local, we have from (iv) that $\dim (R_P)\hat{} = \dim (R_P)^*$, as required. □

2. Pseudoaffine rings

From now on we are concerned with an algebra R over a noetherian ring D such that $R \otimes_D k(p)$ is an affine ring over $k(p)$ for every prime ideal p of D. For convenience sake we call such an R a pseudoaffine ring over D. In this section we collect some basic properties of pseudoaffine rings. We begin by showing that a pseudoaffine ring over D is not necessarily an affine ring over D.

Example 2.1. Let $D = \mathbb{Z}_{(p)}$ and let $R = \mathbb{Z}_{(p)} + X\mathbb{Q}[X]$, where p is a prime number and X is an indeterminate. Then D is a discrete valuation ring, and there exist only two fibres. It is easy to check that the closed fibre $R \otimes_D k(p)$ is isomorphic to $k(p)$ and the generic fibre $R \otimes_D k(0)$ is isomorphic to $k(0)[X]$. Hence R is a pseudoaffine ring over D, while R is not an affine ring over D.

Notice that $\mathrm{tr.deg}_{k(p)} R \otimes_D k(p) = 0$ and $\mathrm{tr.deg}_{k(0)} R \otimes_D k(0) = 1$ in this example. This gives rise to the following:

Definition 2.2. Let D be a noetherian ring and let R be a D-algebra. For a given positive integer n, we say that R is a *pseudoaffine ring of rank* n over D if $R \otimes_D k(p)$ is an affine domain over $k(p)$ such that $\mathrm{tr.deg}_{k(p)} R \otimes_D k(p) = n$ for every $p \in \mathrm{Spec}(D)$.

Hereafter we assume that D is a noetherian ring and R is a flat pseudoaffine ring of rank n over D, i.e., a flat D-algebra which is a pseudoaffine ring of rank n over D.

Lemma 2.3. (i) *Let S be a multiplicatively closed subset of D. Then $R \otimes_D S^{-1}D$ is a flat pseudoaffine ring of rank n over $S^{-1}D$.*

(ii) *Let q be an ideal of D. Then $R \otimes_D D/q$ is a flat pseudoaffine ring of rank n over D/q.*

(iii) *Let x be a regular element of R such that $x \notin mR$ for every $m \in \operatorname{Max}(D)$. Then $R[x^{-1}]$ is a flat pseudoaffine ring of rank n over D.* □

This lemma is easily proved and we omit the proof.

Lemma 2.4. (i) *R is faithfully flat over D.*

(ii) *If p is a prime ideal of D, then pR is a prime ideal of R.*

(iii) *If D is an integral domain, then R is an integral domain.*

(iv) *If P is a prime ideal of R, then PR_P is a finitely generated ideal of R_P.*

Proof. (i) For every $m \in \operatorname{Max}(D)$, we have $\operatorname{tr.deg}_{D/m} R/mR = n$, and hence $mR \neq R$. Since R is flat over D, this shows that R is faithfully flat over D.

(ii) Notice that $R/pR \cong R \otimes_D D/p$ is flat over D/p. Hence, letting $S = (D/p) \setminus \{0\}$, we obtain $R \otimes_D k(p) \cong S^{-1}(R/pR) \supset R/pR$. Since $R \otimes_D k(p)$ is an integral domain, from this it follows that $pR \in \operatorname{Spec}(R)$, as desired.

(iii) This is an immediate consequence of (ii).

(iv) Let $p = P \cap D$. Then $R_p/pR_p \cong R \otimes_D k(p)$ is an affine domain over $k(p)$. Hence PR_p/pR_p is a finitely generated ideal, and so is PR_p. Thus PR_P is finitely generated, too. □

Lemma 2.5. *If* $\dim D = 0$, *then R is an affine ring over D.*

Proof. Let $p_1, \ldots, p_r$ be all the prime ideals of D. Then $p_1 R \cap \cdots \cap p_r R = (p_1 \cap \cdots \cap p_r)R$ is a finitely generated nilpotent ideal of R. Hence, by Lemma 2.3(ii) and Lemma 1.1(i), we may assume that D is a field. Then the assertion is obvious. □

Lemma 2.6. *Suppose that D is normal and that $R \otimes_D k(p)$ is normal for every $p \in \operatorname{Spec}(D)$. Then R is normal.*

Proof. By Lemma 2.3(i), it suffices to prove the assertion in the case where D is a normal local domain. In this case R is an integral domain by Lemma 2.4(iii). We use the induction on $d = \dim D$. There is nothing to prove when $d = 0$. If $d = 1$, then D is a discrete valuation ring and, letting m be the maximal ideal of D, we have $m = xD$ for some $x \in D$. Note that x is a prime element of R by virtue of Lemma 2.4(ii). Since $R[x^{-1}] \cong R \otimes_D k(0)$ is normal, Lemma 1.2(i) then implies that R is normal. Then it follows from Lemma 2.3(i) and the induction hypothesis that $R[a^{-1}]$ is normal. Furthermore, since D is normal, we have $\dim D_p = 1$ for every $p \in \operatorname{Ass}(D/aD)$. Hence, by Lemma 2.3(i) and by what we have proved, R_p is normal for every $p \in \operatorname{Ass}(D/aD)$. Then one can apply Lemma 1.2(ii) to conclude that R is normal, and the proof is completed. □

Lemma 2.7. (i) *For every* $P \in \mathrm{Spec}(R)$, *we have*

$$\mathrm{ht}(P) = \mathrm{ht}(p) + n - \mathrm{tr.deg}_{D/p} R/P,$$

where $p = P \cap D$. *In particular*, $\mathrm{ht}(pR) = \mathrm{ht}(p)$ *for* $p \in \mathrm{Spec}(D)$.

(ii) *If* $\dim D$ *is finite, then* $\dim R = \dim D + n$.

Proof. (i) The general case follows from the case where D is an integral domain, and we assume that D is an integral domain. Then R is also an integral domain by Lemma 2.4(iii). Notice that we may regard D as a subring of R by virtue of Lemma 2.4(i). Hence, by [2, Theorem 1], we obtain

$$\mathrm{ht}(P) \leq \mathrm{ht}(p) + \mathrm{tr.deg}_D R - \mathrm{tr.deg}_{D/p} R/P. \tag{1}$$

In particular, applying (1) with $P = pR$ and noting that $\mathrm{tr.deg}_D R = \mathrm{tr.deg}_{D/p} R/pR = n$, we have $\mathrm{ht}(pR) \leq \mathrm{ht}(p)$. On the other hand, it follows from Lemma 2.4 that $\mathrm{ht}(pR) \geq \mathrm{ht}(p)$. Hence we get $\mathrm{ht}(pR) = \mathrm{ht}(p)$ for $p \in \mathrm{Spec}(D)$, which proves the second assertion. Now we will show that the equality generally holds in (1). For this purpose, replacing D and R by D_p and R_p, respectively, we may suppose that D is a local domain with maximal ideal p. Then $R/pR \cong R \otimes_D D/p$ is an affine domain over the residue field D/p, and hence we have

$$\mathrm{ht}(P/pR) = \mathrm{tr.deg}_{D/p} R/pR - \mathrm{tr.deg}_{D/p} R/P.$$

Since $\mathrm{ht}(P) \geq \mathrm{ht}(P/pR) + \mathrm{ht}(pR) = \mathrm{ht}(P/pR) + \mathrm{ht}(p)$, it follows from this equality that

$$\mathrm{ht}(P) \geq \mathrm{ht}(p) + \mathrm{tr.deg}_{D/p} R/pR - \mathrm{tr.deg}_{D/p} R/P. \tag{2}$$

Comparing (1) and (2), we infer that the equality does hold in (1).

(ii) Let M be a maximal ideal of R and let $m = M \cap D$. Then, by virtue of (i), we have $\mathrm{ht}(M) \leq \mathrm{ht}(m) + n$, which implies that $\dim R \leq \dim D + n$. Conversely, let m be a maximal ideal of D and take an $M \in \mathrm{Max}(R)$ with $mR \subset M$. Then R/mR is an affine domain over D/m, and so

$$\mathrm{tr.deg}_{D/m} R/M = \mathrm{tr.deg}_{D/m} (R/mR)/(M/mR) = 0.$$

It then follows from (i) that $\mathrm{ht}(M) = \mathrm{ht}(m) + n$. Hence we have $\dim R \geq \dim D + n$, which completes the proof. □

3. Ideal-adic completion of a pseudoaffine ring

In this section we assume that (D, m) is a noetherian local domain which is catenary and R is a flat pseudoaffine ring of rank n over D. For a D-module M, we denote by M^* the m-adic completion of M. The purpose of this section is to prove Lemmas 3.2 and 3.3 below, which we need in the last section.

Lemma 3.1. *Let P be a prime ideal of R and let q be a prime ideal of D such that $q \subset P$ and* $\mathrm{ht}(q) = 1$. *Then* $\mathrm{ht}(P/qR) = \mathrm{ht}(P) - 1$.

Proof. Let $p = P \cap D$. Then, by Lemma 2.7(i), we have

$$\mathrm{ht}(P) = \mathrm{ht}(p) + n - \mathrm{tr.deg}_{D/p} R/P. \tag{3}$$

On the other hand, since R/qR is a flat pseudoaffine ring of rank n over D/q (cf. Lemma 2.3(ii)), again by Lemma 2.7(i), we have

$$\mathrm{ht}(P/pR) = \mathrm{ht}(p/q) + n - \mathrm{tr.deg}_{D/p} R/P. \tag{4}$$

Combining (3) and (4), we obtain

$$\mathrm{ht}(P) - \mathrm{ht}(P/qR) = \mathrm{ht}(p) - \mathrm{ht}(p/q) = 1$$

because D is catenary and $\mathrm{ht}(q) = 1$. This completes the proof. □

Lemma 3.2. *Let S be a multiplicatively closed subset of R such that $S \cap mR = \emptyset$. Then* $\dim (S^{-1}R)^* \geq \dim S^{-1}R$. *In particular*, $\dim R^* \geq \dim R$.

Proof. We use the induction on $d = \dim D$. If $d = 0$, then the assertion is obvious. Suppose that $d > 0$ and let q be a prime ideal of D with $\mathrm{ht}(q) = 1$. Then, noting that $S^{-1}R$ is flat over D, it follows from Lemma 1.3(ii) that $(S^{-1}R)^*/q(S^{-1}R)^* \cong (S^{-1}R/q(S^{-1}R))^*$. Let $\bar{S}$ be the image of S by the natural map $R \to R/qR$. Then $S^{-1}R/q(S^{-1}R) \cong \bar{S}^{-1}(R/qR)$ and therefore, by Lemma 2.3(ii) and induction hypothesis, we have

$$\dim (S^{-1}R)^*/q(S^{-1}R)^* \geq \dim S^{-1}R/q(S^{-1}R).$$

On the other hand, Lemma 3.1 implies that

$$\dim S^{-1}R/q(S^{-1}R) = \dim S^{-1}R - 1.$$

Notice that if a is a non-zero element of D, then a is a regular element of R^* (cf. Lemma 1.3(i)). Hence we obtain

$$\begin{aligned}\dim (S^{-1}R)^* &\geq \dim (S^{-1}R)^*/q(S^{-1}R)^* + 1\\ &\geq \dim (S^{-1}R)/q(S^{-1}R) + 1\\ &= \dim S^{-1}R,\end{aligned}$$

as desired. □

Lemma 3.3. *Let P be a prime of R. Then* $\dim (R_P)^\wedge \geq \dim R_P$.

Proof. Letting $p = P \cap D$, we may suppose that $p = m$; replace D and R by D_p and R_p, respectively, if necessary. Then $mR \subset P$ and $R/mR \cong R \otimes_D D/m$ is noetherian. Hence, by Lemma 1.3(v), we have $\dim (R_p)^\wedge = \dim (R_p)^*$. Our assertion then follows from the preceding lemma. □

4. Main theorem

In this section, by making use of the results obtained in the previous sections, we prove the main theorem of this paper. In the following, unless explicitly stated, we assume that D is an excellent normal semi-local domain and R is a flat pseudoaffine ring of rank n over D. Then we may regard D as a subring of R and R is an integral domain (cf. Lemma 2.4).

Lemma 4.1. *Let P be a prime ideal of R. If R is normal, then R_P is a locality over D, i.e., there exist an affine domain B over D and a prime ideal Q of B such that $R_P = B_Q$.*

Proof. Let $m = P \cap D$. Then, replacing D and R by D_m and R_m, respectively, we may suppose that D is a local ring with maximal ideal m. Let $k = D/m$. Then $R/mR \cong R \otimes_D k$ is an affine domain over k, so that $R/P \cong (R/mR)/(P/mR)$ is also an affine domain over k. Hence the residue field $R_P/PR_P \cong Q(R/P)$ is finitely generated over k, and we can choose elements $x_1, \ldots, x_r$ of R such that $R_P/PR_P = k(\bar{x}_1, \ldots, \bar{x}_r)$, where $\bar{x}_i$ denotes the residue class of x_i in R_P/PR_P. Let $K = Q(D)$ and $L = Q(R)$. Then L is finitely generated over K since $R \otimes_D K$ is an affine domain over K. Hence we have $L = K(y_1, \ldots, y_s)$ for some $y_1, \ldots, y_s \in R$. Finally, by Lemma 2.4(iv), there exist elements $z_1, \ldots, z_t$ of R such that $PR_P = (z_1, \ldots, z_t)R_P$. Let $A = D[x_1, \ldots, x_r, y_1, \ldots, y_s, z_1, \ldots, z_t]$ and let A' be the integral closure of A in $Q(A) = L$. Then we have $A' \subset R$ because $A \subset R$ and R is normal. Moreover, since D is excellent, so is A and hence A' is a finite A-module. Consequently, A' is an affine domain over D. Let $T = A'_Q$, where $Q = P \cap A'$, and let $\mathfrak{n} = QA'_Q$. Then, by the construction, it is obvious that $(T, \mathfrak{n})$ is a normal locality over D such that (a) $D \subset T \subset R_P$, that (b) $Q(T) = Q(R_P)$, that (c) $T/\mathfrak{n} = R_P/RP_P$ and that (d) $\mathfrak{n}R_P = PR_P$. Furthermore we have $\dim T = \dim R_P$. In fact, it follows from Lemma 2.7(i) that

$$\dim R_P = \dim D + n - \text{tr.deg}_k\, R_P/PR_P. \tag{5}$$

On the other hand, since D is excellent, the dimension formula holds between D and T, and so

$$\dim T = \dim D + n - \text{tr.deg}_k\, T/\mathfrak{n}. \tag{6}$$

Since $R_P/PR_P = T/\mathfrak{n}$, comparing (5) and (6) we obtain $\dim T = \dim R_P$, as desired.

Now we claim that $T = R_P$. For this purpose it is enough to show that $T^\wedge = (R_P)^\wedge$. Indeed, if this is the case then, for an arbitrary element x of R_P, writing $x = b/a$ with $a, b \in T$, we have

$$b \in aR_P \cap T \subset a(R_P)^\wedge \cap T = aT^\wedge \cap T = aT,$$

and therefore $x = b/a \in T$. This means $T = R_P$, as asserted.

It thus remains to prove $T^\wedge = (R_P)^\wedge$. Notice that T is an excellent normal local domain. Hence $T^\wedge$ is an integral domain by the analytic normality of excellent rings (cf. [3, Theorem 79]). By the condition (d), $(R_P)^\frown$ coincides with the $\mathfrak{n}$-adic completion of R_P. Hence there exists a natural map $f: T^\wedge \to (R_P)^\frown$, which is surjective by virtue of (c). Then, by Lemma 3.3, we have

$$\dim T = \dim T^\wedge \geq \dim f(T^\wedge) = \dim (R_P)^\wedge \geq \dim R_P.$$

Since $\dim T = \dim R_P$, from this it follows that $\dim T^\wedge = \dim f(T^\wedge)$. This shows that f is injective because $T^\wedge$ is an integral domain. Therefore f is an isomorphism, and $T^\wedge = (R_P)^\frown$, as required. $\square$

Lemma 4.2. *If R is normal, then there exists a non-zero element x of R such that $R[x^{-1}]$ is an affine domain over D.*

Proof. Let $m_1, \ldots, m_r$ be all the maximal ideals of D and suppose that, for each $i = 1, \ldots, r$, there is an $x_i \in R$ such that $R_{m_i}[x^{-1}]$ is an affine domain over D_{m_i}. Then, letting $x = x_1 \cdots x_r$, it follows from Lemma 1.1(iii) that $R[x^{-1}]$ is an affine domain over D. Hence, by Lemma 2.3(i), we see that it suffices to prove the assertion in the case where D is a local ring.

Let m be the maximal ideal of D and let $k = D/m$. For a D-module N, we denote by N^* the m-adic completion of N. Since R/mR is an affine domain over k and $\text{tr.deg}_k R/mR = n$, it follows from Noether's normalization lemma (cf. [3, (14.G)]) that there exist elements $z_1, \ldots, z_n$ of R such that their residue classes $\bar{z}_1, \ldots, \bar{z}_n$ in R/mR are algebraically independent over k and that R/mR is integral over $k[\bar{z}_1, \ldots, \bar{z}_n]$. Let $B = D[z_1, \ldots, z_n]$.

We will show that B is isomorphic to a polynomial ring in n-variables over D, i.e., $z_1, \ldots, z_n$ are algebraically independent over D. Let $\mathfrak{M} = mR \cap B$. Then we have $\mathfrak{M} = mB$. Indeed, consider a surjective homomorphism $\psi: D^{[n]} \to B$, where $D^{[n]}$ denotes a polynomial ring in n-variables over D, and let $I = \psi^{-1}(\mathfrak{M})$. Then we have $D^{[n]}/I \cong B/\mathfrak{M} \cong k[\bar{z}_1, \ldots, \bar{z}_n]$. Since $\bar{z}_1, \ldots, \bar{z}_n$ are algebraically independent over k and $mD^{[n]} \subset I$, from this it follows that $I = mD^{[n]}$. Hence $\mathfrak{M} = \psi(I) = mB$, as desired. Consequently B^* and R^* coincide with the $\mathfrak{M}$-adic completions of B and R, respectively. Note that $R^*/\mathfrak{M}R^* = R^*/mR^* \cong R/mR$ by virtue of Lemma 1.3(ii). Since $B^*/\mathfrak{M}B^* \cong B/\mathfrak{M}$, we know that $R^*/\mathfrak{M}R^*$ is a finite $B^*/\mathfrak{M}B^*$-module. It then follows from [7, Chapter VIII, §3, Corollary 2] that R^* is a finite B^*-module. This implies

$$\dim B^* \geq \dim R^*. \tag{7}$$

On the other hand, by Lemma 3.2 and Lemma 2.7(ii), we have

$$\dim R^* \geq \dim R = \dim D + \text{tr.deg}_D R. \tag{8}$$

Moreover, since B is noetherian, it follows that

$$\dim B^* \leq \dim B \leq \dim D + \text{tr.deg}_D B. \tag{9}$$

Combining (7), (8) and (9), we obtain $\text{tr.deg}_D B \geq \text{tr.deg}_D R$, while obviously $\text{tr.deg}_D B \leq \text{tr.deg}_D R$. Hence $\text{tr.deg}_D B = \text{tr.deg}_D R = n$, and our claim is verified.

Let B' be the integral closure of B in $Q(R)$. Then we have $B' \subset R$ and $Q(B') = Q(R)$. Notice that B is an excellent normal domain and that $Q(R)$ is a finite extension field of $Q(B)$. Hence B' is a finite B-module, so that there exist only finitely many prime ideals of B' lying over $\mathfrak{M}$. Clearly $\mathfrak{M}' = mR \cap B'$ is one of them. Let $\mathfrak{M}'_1, \dots, \mathfrak{M}'_r$ be the remaining prime ideals of B' lying over $\mathfrak{M}$. Then $\mathfrak{M}', \mathfrak{M}'_1, \dots, \mathfrak{M}'_r$ are all the minimal prime ideals of $\mathfrak{M}B' = mB'$ because the going down theorem holds between B and B' (cf. [3, Theorem 5]). Take an element t of B' such that $t \in (\mathfrak{M}'_1 \cup \dots \cup \mathfrak{M}'_r) \setminus \mathfrak{M}'$. Let $\bar{B} = B'[t^{-1}]$ and let $\bar{R} = R[t^{-1}]$. Then we have $\bar{B} \subset \bar{R}$ and, by Lemma 2.3(iii), $\bar{R}$ is a flat pseudoaffine ring of rank n over D.

We now assert that $\bar{B}^* = \bar{R}^*$. Let $\bar{\mathfrak{M}} = \mathfrak{M}'\bar{B}$. Then $\bar{\mathfrak{M}}$ is the unique minimal prime ideal of $m\bar{B} = \mathfrak{M}\bar{B}$ by the choice of t. Hence $\bar{B}^*$ coincides with the $\bar{\mathfrak{M}}$-adic completion of $\bar{B}$. On the other hand, $\bar{R}$ also coincides with the $\bar{\mathfrak{M}}$-adic completion of $\bar{R}$ since $\bar{\mathfrak{M}}\bar{R} = m\bar{R}$. Notice that R/mR is a finite $B'/\mathfrak{M}'$-module because $B/\mathfrak{M} \subset B'/\mathfrak{M}' \subset R/mR$ and R/mR is a finite $B/\mathfrak{M}$-module. Hence $\bar{R}/\bar{\mathfrak{M}}\bar{R}$ is a finite $\bar{B}/\bar{\mathfrak{M}}$-module, and we know that $\bar{R}^*$ is a finite $\bar{B}^*$-module. Therefore, letting f be the natural map $\bar{B}^* \to \bar{R}^*$, it follows from Lemma 3.2 that

$$\dim \bar{B} \geq \dim \bar{B}^* \geq \dim f(\bar{B}^*) = \dim \bar{R}^* \geq \bar{R}. \tag{10}$$

On the other hand, by Lemma 2.7(ii), we have

$$\dim \bar{R} = \dim D + n = \dim \bar{B}. \tag{11}$$

From (10) and (11) we obtain

$$\dim \bar{B}^* = \dim f(\bar{B}^*) \tag{12}$$

Notice that $\bar{B}$ is an excellent normal domain and $\bar{\mathfrak{M}}$ is a prime ideal of $\bar{B}$. Hence $\bar{B}^*$ is a normal domain by virtue of Lemma 4.3 below. Together with (12) this implies that f is injective. Look at the following commutative diagram:

$$\begin{tikzcd}
\bar{B} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \bar{R} \arrow[d, "g"] \\
\bar{B}^* \arrow[r, "f"] & \bar{R}^*
\end{tikzcd}$$

where α, β and g are natural maps. Since α, β and f are injective and $Q(\bar{B}) = Q(\bar{R})$, it is readily shown that g is also injective. Let $y_1, \dots, y_s$ be elements of $\bar{R}$ such that their residue classes in $\bar{R}/\bar{\mathfrak{M}}\bar{R}$ generate $\bar{R}/\bar{\mathfrak{M}}\bar{R}$ over $\bar{B}/\bar{\mathfrak{M}}$. Then $\bar{R}^*$ is generated by $g(y_1), \dots, g(y_s)$ over $\bar{B}^*$ (cf. [7, Chapter VIII, §3, Corollary 2]). Therefore we have $\bar{R}^* \subset Q(\bar{B}^*)$ because $Q(\bar{B}) = Q(\bar{R})$ and g is injective. Since $\bar{B}^*$ is normal and $\bar{R}^*$ is integral over $\bar{B}^*$, from this it follows that $\bar{B}^* = \bar{R}^*$, as asserted.

We next claim that if $s \in D$ and if $\mathfrak{P} \in \operatorname{Ass}(\bar{B}/s\bar{B})$, then $\mathfrak{P} \subset \bar{\mathfrak{M}}$. The assertion being trivial in the case $s = 0$, we assume that $s \neq 0$. Let $\mathfrak{P}' = \mathfrak{P} \cap B'$. Then $\mathfrak{P}' \in \operatorname{Ass}(B'/sB')$, which implies that $\operatorname{ht}(\mathfrak{P}') = 1$ since B' is normal. Notice that the dimension formula holds between B and B' because B is excellent. Hence, letting $\mathfrak{P}=\mathfrak{P}'\cap B$, we have $\mathrm{ht}(\mathfrak{P})=1$, so that $\mathfrak{P}\in \mathrm{Ass}(B/sB)$. Since $\mathrm{Ass}(B/sB)=\{pB \mid p\in \mathrm{Ass}(D/sD)\}$ (recall that B is isomorphic to a polynomial ring over D), this yields $\mathfrak{P}\subset mB=\mathfrak{M}$, and it follows that $\bar{\mathfrak{P}}\bar{B}_{\mathfrak{M}}\in \mathrm{Spec}(\bar{B}_{\mathfrak{M}})$. By the choice of t, $\bar{B}_{\mathfrak{M}}$ is a local ring with maximal ideal $\bar{\mathfrak{M}}\bar{B}_{\mathfrak{M}}$, and therefore we have $\bar{\mathfrak{P}}\bar{B}_{\mathfrak{M}}\subset\bar{\mathfrak{M}}\bar{B}_{\mathfrak{M}}$, which implies $\bar{\mathfrak{P}}\subset\bar{\mathfrak{M}}$, as required.

It follows from this and Krull's intersection theorem that $s\bar{B}^*\cap\bar{B}=s\bar{B}$ for every $s\in D$. Hence, if we set $S=D\setminus\{0\}$, then we get $S^{-1}\bar{B}\cap\bar{R}=\bar{B}$. Indeed, for every $w\in S^{-1}\bar{B}\cap\bar{R}$, writing $w=b/s$ with $b\in\bar{B}$ and $s\in S$, we have

$$b\in s\bar{R}\cap\bar{B}\subset s\bar{R}^*\cap\bar{B}=s\bar{B}^*\cap\bar{B}=s\bar{B},$$

and hence $w=b/s\in\bar{B}$. Recall that $\bar{R}$ is a pseudoaffine ring over D. Hence $S^{-1}\bar{R}\cong\bar{R}\otimes_D S^{-1}D$ is an affine domain over $S^{-1}D$, and so is $S^{-1}\bar{R}$ over $S^{-1}\bar{B}$. Furthermore we have $Q(S^{-1}\bar{B})=Q(S^{-1}\bar{R})$ by the construction of $\bar{B}$. Hence there exists an element u of B' such that $S^{-1}\bar{B}[u^{-1}]=S^{-1}\bar{R}[u^{-1}]$. Then it is obvious that

$$S^{-1}\bar{B}[u^{-1}]\cap\bar{R}[u^{-1}]=\bar{R}[u^{-1}]. \tag{13}$$

On the other hand, since $S^{-1}\bar{B}\cap\bar{R}=\bar{B}$, we have

$$S^{-1}\bar{B}[u^{-1}]\cap\bar{R}[u^{-1}]=\bar{B}[u^{-1}]. \tag{14}$$

From (13) and (14) we obtain $\bar{B}[u^{-1}]=\bar{R}[u^{-1}]$, which shows that $\bar{R}[u^{-1}]$ is an affine domain over D. Hence, letting $x=tu$, we know that x satisfies the required condition. □

Lemma 4.3. *Let A be an excellent normal domain and let $\mathfrak{P}$ be a prime ideal of A. Then the $\mathfrak{P}$-adic completion of A is a normal domain.*

Proof. We denote by A^* the $\mathfrak{P}$-adic completion of A. Then A^* is normal since A is an excellent normal ring. Hence it suffices to show that A^* is an integral domain. Let $\mathfrak{P}^{(n)}=\mathfrak{P}^nA_{\mathfrak{P}}\cap A$ for a positive integer n. Then there exists a natural injection $\operatorname{proj\,lim} A/\mathfrak{P}^{(n)}\to(A_{\mathfrak{P}})\hat{}$. On the other hand, it follows from [6, Theorem 1] that $\operatorname{proj\,lim} A/\mathfrak{P}^{(n)}$ coincides with A^*. Hence we have $A^*\subset(A_{\mathfrak{P}})\hat{}$. Since $(A_{\mathfrak{P}})\hat{}$ is an integral domain, we know that A^* is also an integral domain. □

We are now ready to establish the following:

Theorem 4.4. *Let D be an excellent normal semi-local domain and let R be a flat D-algebra satisfying the following conditions:*

(i) *$R\otimes_D k(p)$ is a normal affine domain over $k(p)$ for every $p\in\mathrm{Spec}(D)$;*

(ii) *$\mathrm{tr.deg}_{k(p)} R\otimes_D k(p)$ is constant independent of $p\in\mathrm{Spec}(D)$.*

Then R is a normal affine domain over D.

Proof. Our assertion follows from Lemmas 2.6, 4.1, 4.2 and [5, Theorem 2.20]. □

As mentioned in the introduction this theorem has an application to the theory of pseudopolynomial rings. Recall that an algebra A over a field K is said to be geometrically normal (resp. geometrically irreducible) over K if $A\otimes_K L$ is a normal ring (resp. an integral domain) for every finite extension field L of K.

Corollary 4.5. *Let D be an excellent semi-local ring and R a flat pseudoaffine ring of finite rank over D. Suppose that $R\otimes_D k(p)$ is geometrically normal and irreducible over $k(p)$ for every $p\in\operatorname{Spec}(D)$. Then R is an affine ring over D.*

Proof. It suffices to prove the assertion in the case where D is an integral domain (cf. the proof of Lemma 2.5). Let D' be the integral closure of D in $Q(D)$ and let $R'=R\otimes_D D'$. Then, for every $p'\in\operatorname{Spec}(D')$, letting $p=p'\cap D$, we have

$$\begin{aligned} R'\otimes_{D'} k(p') &\cong (R\otimes_D D')\otimes_{D'} k(p) \\ &\cong (R\otimes_D k(p))\otimes_{k(p)} k(p'). \end{aligned} \tag{15}$$

Note that $k(p')$ is a finite extension field of $k(p)$ because D' is a finite D-module. Since $R\otimes_D k(p)$ is geometrically normal and irreducible over $k(p)$, we know from (15) that $R'\otimes_{D'} k(p')$ is a normal affine domain over $k(p')$ such that $\operatorname{tr.deg}_{k(p')} R'\otimes_{D'} k(p')$ is constant independent of p'. We can then apply Theorem 4.4 to conclude that R' is an affine ring over D'. Since D' is a finite D-module, it follows that R' is an affine ring over D. Let $\bar{R}=R[D']$. Then there exists a surjective D-algebra map $R'\to\bar{R}$, and hence $\bar{R}$ is also an affine ring over D. Since $R\subset\bar{R}$ and $\bar{R}$ is a finite R-module, we know from [4, Lemma 3.1] that R is an affine ring over D. □

Corollary 4.6. *Let D be a noetherian semi-local ring and let R be a flat pseudopolynomial D-algebra. Then R is an affine ring over D.*

Proof. It is easy to check that $R\otimes_D\hat{D}$ is a flat pseudopolynomial $\hat{D}$-algebra. Hence, by Lemma 1.1(ii), we may assume that D is complete. Since R is a pseudopolynomial ring, it is obvious that each fibre ring is geometrically normal and irreducible over its base field. A complete semi-local ring is excellent, and hence the assertion follows from the preceding corollary. □

We conclude this paper by examining the conditions stated in Theorem 4.4.

Remark 4.7. (i) If D is not semi-local, then Theorem 4.4 does not hold in general. For example, let $D=\mathbb{Z}$ and let $R=\mathbb{Z}[X,X/2,\dots,X/p,\dots]$, where X is an indeterminate and p runs through all the prime numbers. Then, as is easily seen, R is a flat pseudopolynomial D-algebra in one variable, however R is not an affine ring over D.

(ii) We can not remove the assumption that R is flat over D from Theorem 4.4. As for the example, see [1, p. 114].

(iii) Example 2.1 shows that the assumption (ii) is needed in Theorem 4.4.

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